① Course info

mathweb.ucsd.edu/~lni/math251B

Two important links.

schedule.html — progress & readings

homework.html — works

Grading: homework or presentation

in writting & presentation.

② Lie groups & Lie algebras.

Def 1.a: G is a Lie group if G is a smooth mfld. & a group satisfying

$$\varphi: G \to G$$
$$g \to \varphi(g) = g^{-1}$$

& $\psi: G \times G \to G$

$$(g,\ h) \quad \psi(g, h) = g \cdot h$$

are smooth.

Remark: (i) $\exists$ unique analytic structure

(ii) $\exists$ Hilbert's 5th problem.

Ref: Potrjangin's book & Montgomery & Zippin's book.

E.g. (i) $GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ open subset $A^{-1} = \frac{1}{|A|} \operatorname{adj} A$

or $\mathbb{C}$

(ii) $SL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$

$A \quad \det(A) = 1$

Defn 1.6 Lie algebra $\mathfrak{g}$. (over $\mathbb{R}$, $\mathbb{C}$ or other fields)

(i) a vector space over $\mathbb{F}$ - field. (focus is on $\mathbb{C}$)

For number theory needs $\mathbb{Z}_p$

(ii) $\forall\, v, w \in \mathfrak{g}$ $\exists\, [v, w]$ bilinear in v & w $\quad \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $[\cdot, \cdot] \mapsto$

& satisfying: (a) $[v, w] = -[w, v]$

(b) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

Jacobi identity

E.g. (i) $(\mathbb{R}^3, \times)$

namely $[x, y] \doteqdot x \times y$ cross product.

$$\begin{bmatrix} (x \times y) \times z \\ + (y \times z) \times x \\ + (z \times x) \times y = 0 \end{bmatrix}$$

Ex: Check it! which Lie group has it as its Lie algebra?

Hint: use quaternions!

(ii) $\mathfrak{g} = \mathcal{X}(M)$ - Smooth vector fields on M.

$X = X^i \frac{\partial}{\partial x^i}$ locally $\quad$ Vector valued functions $\quad X^i(x) \frac{\partial}{\partial x^i}$ $\quad \begin{pmatrix} X^1(x) \\ \vdots \\ X^n(x) \end{pmatrix}$

$[X, Y] f = XYf - YXf$

$$X^i \frac{\partial}{\partial x^i}\left(Y^k \frac{\partial f}{\partial x^k}\right) - Y^i \frac{\partial}{\partial x^i}\left(X^k \frac{\partial f}{\partial x^k}\right)$$

$$= \left(X^i \frac{\partial Y^k}{\partial x^i}\right) \frac{\partial f}{\partial x^k} - \left(Y^i \frac{\partial X^k}{\partial x^i}\right) \frac{\partial}{\partial x^k}$$

Namely $[X\ Y]^k = \left(\sum_i X^i \frac{\partial Y^k}{\partial x^i} - \sum_i Y^i \frac{\partial X^k}{\partial x^i} \right)$ $k=1,\cdots,n$

250A/251A $\Rightarrow$ $\mathcal{X}(M)$ is a Lie algebra!

③ Relation: — via the differentiation

(i). $x, y \in G_e$ $= \mathfrak{g}$ — the tangent space at $e \in G$ (identity).

G_e — The tangent space at e

$[x, y] := [X, Y]|_e$ $\qquad G_e \to G_g$

$$\boxed{X(g) := (L_g)_*(x)}$$

$L_g: G \to G$
$h \to gh$.

Such a vector field called the left invariant v.f.

Since $\forall \sigma$

$$\boxed{(L_g)_*(X(\sigma)) = X(g\sigma)}$$

the defining property of a left invariant vector field.

$X(\sigma) = (L_\sigma)_*(x)$, $X(g\sigma) = (L_{g\sigma})_*(x)$

Inner automorphism: $a_h: G \to G$, $a_h(g) = hgh^{-1}$ $\quad \forall h \in G$

$a_h(g_1 g_2) = a_h(g_1) \cdot a_h(g_2)$ a automorphism

$d(a_h)$ or $(a_h)_*: G_e \to G_e$ $\qquad a_{h_1h_2} = a_{h_1} \cdot a_{h_2}$, $\quad a_h \circ a_{h^{-1}} = id$

$G_e = T_e G \simeq \mathfrak{g}$

$df \doteq (f)_*$ ← namely two ways of notation.

is called the adjoint representation of G. $\qquad n = \dim G$

$Ad(h): \mathfrak{g} \to \mathfrak{g}$ $\qquad Ad: G \to GL(n, \mathfrak{g})$ is a representation

$Ad(h) := (a_h)_*$ $\qquad a_h \cdot a_{h^{-1}} = id$

$Ad(h_1h_2) = d\, a_{h_1h_2} = d(a_{h_1} \cdot a_{h_2}) = da_{h_1} \cdot da_{h_2} = Ad(h_1)\, Ad(h_2)$

Clearly. $Ad(h_1 h_2) = Ad(h_1) \cdot Ad(h_2)$

[$R: G_1 \to G_2$ is a homomorphism.
if $R(g_1 g_2) = R(g_1) \cdot R(g_2)$.]

A homomorphism $\rho: G \to GL(n, V)$ is called a linear representation of G.

A linear vector space V.

[namely a realization of G into $GL(n, V)$]

$\chi_\rho(g) := tr(\rho(g))$ is called the character (function).

$Ad: G \to GL(n, \mathfrak{g})$

Lemma: $[X, Y](e) = \frac{d}{dt}\Big|_{t=0} Ad(\exp(tX))(Y_e)$

$d(a_{\exp(tX)})(Y_e)$

X, Y left invariant v.f.

Here $\exp(tX)$ denote the 1 parameter family of diffeomorphisms generated by X with $0 \to e$ $\boxed{\varphi_t(e)}$

Let $\varphi_t(a)$ be the diff generated by X with $\varphi_0(a) = a$

$\varphi_t(a) = a \cdot \underset{\underset{\exp(tX)}{\parallel}}{\varphi_t(e)}$

$\frac{d}{dt}(\varphi_t(a)) = X(\varphi_t(a))$ $\quad t = 0 \to a \quad \Downarrow$

since $(a\, \varphi_t(e))^{\cdot} = (L_a)_*(\dot{\varphi}_t)$

$\frac{d}{dt}\Big|(a\, \varphi_t(e)) = (L_a)_*(X(\varphi_t(e)))$

$=$ Left invariant $X(a \cdot \varphi_t(e))$

Hence $\varphi_t(p) = R_{\exp(tX)}(p)$

Right multiplication.

Use $[X, Y](e) = \frac{d}{dt}\Big|_{t=0}\left[d\varphi_{-t}\left(Y_{\varphi_t(e)}\right)\right]$ holds for any manifold.

$\mathcal{L}_X Y$:=

by definition

$= \frac{d}{dt}\left(dR_{\exp(-tX)}\, dL_{\exp(tX)}(Y_e)\right)$

$= \frac{d}{dt}\left[Ad(\exp(tX))(Y_e)\right]$

$a_{\exp(tX)}(g) = \exp(tX)\, g \exp(-tX)$ □

(ii) In local coordinates near e - $x(e) = 0$

$$\psi(x, 0) = \psi(0, x) = x$$

$$\boxed{x \cdot e = e \cdot x = x}$$

$$\psi(x, \varphi(x)) = 0$$

$$x \cdot x^{-1} = e$$

$$\psi(x, \psi(y, z)) = \psi(\psi(x, y), z)$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

$$\Rightarrow \quad \psi^\alpha(x, y) = \boxed{x^\alpha + y^\alpha} + \boxed{b^\alpha_{\beta\gamma} x^\beta y^\gamma} + \underset{\text{higher order.}}{O(3)}$$

$2 \dfrac{\partial^2 \psi^\alpha}{\partial x^\beta \partial y^\gamma} (0,0)$

still commutative if $b^\alpha_{\beta\gamma} = b^\alpha_{\gamma\beta}$

no terms $x^\beta x^\gamma$ since $\psi(x, 0) = x$

Define $[\xi, \eta]^\alpha \doteq (\underbrace{b^\alpha_{\beta\gamma} - b^\alpha_{\gamma\beta}}_{C^\alpha_{\beta\gamma}}) \xi^\beta \eta^\gamma = \sum_{\beta,\gamma} C^\alpha_{\beta\gamma} \xi^\beta \eta^\gamma$

measures the commutativeness up to 2nd order.

$\xi = \xi^\alpha \dfrac{\partial}{\partial x^\alpha}, \quad \eta = \eta^\alpha \dfrac{\partial}{\partial x^\alpha}$ locally.

We check that it does satisfy the Jacobi identity.

$\{C^\alpha_{\beta\gamma}\}$ Called the structure Constant

$x \cdot (y \cdot z)$ $\quad (x \cdot y) \cdot z$

Use $\psi(x, \psi(y, z)) = \psi(\underbrace{\psi(x, y)}, z)$

$$= \psi^\alpha(x, y) + z^\alpha + b^\alpha_{\beta\gamma} \psi^\beta z^\gamma$$

$$\| \quad x^\alpha + \psi^\alpha(y, z) + b^\alpha_{\beta\gamma} x^\beta \psi^\gamma(y, z)$$

$$= x^\alpha + y^\alpha + z^\alpha + b^\alpha_{\beta\gamma} x^\beta y^\gamma + \underbrace{b^\alpha_{\beta\gamma} b^\beta_{\mu\nu} x^\mu y^\nu z^\gamma}_{b^\alpha_{\beta s} b^\beta_{\mu\nu}}$$

$$\| \quad x^\alpha + y^\alpha + z^\alpha + b^\alpha_{\beta\gamma} y^\beta z^\gamma + b^\alpha_{\beta\gamma} b^\gamma_{\mu\nu} \underline{x^\beta y^\mu z^\nu}_{x^\mu y^\nu z^s}$$

$b^\alpha_{\mu\gamma} b^\gamma_{\nu s}$

$$\boxed{b^\alpha_{\beta s} b^\beta_{\mu\nu} = b^\alpha_{\mu\beta} b^\beta_{\nu s}} \qquad (*)$$

(*) implies the Jacobi identity.

In fact, (*) is NOT completely correct. The full details will be in Remark (iv). The checking is in scanned hand-written notes

$\{ C^{\alpha}_{\beta\gamma} \doteq b^{\alpha}_{\beta\gamma} - b^{\alpha}_{\gamma\beta} \}$ are called the structure constants

Relation 2: — Integration

Theorem. Let G be a real analytic Lie group. Then in the neighborhood of $e \in G$, the structure constants determine the group multiplication. In fact, we construct ψ. out of $\{C^{\alpha}_{\beta\gamma}\}$

4 Remarks.

ad_X: is a linear map: $\mathfrak{g} \to \mathfrak{g}$, $Y \to [X, Y]$

$= d(Ad)(X)$

(i) Lemma $\Rightarrow$ $ad_X(Y) = [X, Y]$

It is also defined as

$$\boxed{ad_X := (Ad)_*(X)}:$$

$$\boxed{Ad: G \to GL(n, \mathfrak{g})}$$

$(Ad_*): G_e \simeq \mathfrak{g} \longrightarrow gl(n, \mathfrak{g}) \leftarrow$ the Lie algebra of $GL(n, \mathfrak{g})$

Corollary: $Ad_G(\exp(tX)) = \exp(t\, ad(X))$ (**)

pf: In general, if $\rho: G_1 \to G_2$ is a homomorphism i.e. $\rho(g_1 g_2) = \rho(g_1)\cdot\rho(g_2)$

$$\rho(\exp(tX)) = \exp(t\, d\rho(X)) \qquad (**')$$

$\left[\frac{d}{dt}\Big|_{t=t_0} \exp(tX) = dL_{\exp(t_0 X)} \cdot X \right.$

$\left. X(a) = (L_a)_*(X_e) \right]$

1-parameter family of group with the same initial data.

At $t=0$, its tangent is $d\rho(X)$ is the tangent vector.

$Y(\rho(a)) := d\rho(X)(\rho(a)) = d\rho(X(a)) = d\rho((L_a)_*(X_e))$

$$= (L_{\rho(a)})_* \, d\rho(X_e)$$

$$= (L_{\rho(a)})_* Y_e$$

$\rho(ag) = \rho(a)\rho(g)$

X is left invariant $\Rightarrow Y = d\rho(X)$ is left invariant.

Then (**') follows from the uniqueness.

Namely (**) follows from (**') which is more general.

Details, Warner. P104 Thm 3.32.

(ii) The coordinates of 1st kind. & two definitions are the same.

$\exp : G_e \to G \qquad \exp(X) \in G$

namely $\exp(tX)\big|_{t=1}$

$X = \sum x^i e_i$

This is a diffeomorphism $\Rightarrow$ if $\delta \ll 1 \qquad |X| \le \delta$

$\exp : \{|X| \le \delta\} \subset G_e \longrightarrow G$ is a coordinate chart.

(iii) $x(\tau) = X_\tau$ — are the 1-parameter families

$y(t) = Y_t \qquad x(\tau)^{-1} = -X_\tau$

$\boxed{\text{two Lie-bracket definitions coincide}}$

$I = x(\tau)\, y(t)\, x(\tau)^{-1}$

$= \underbrace{\tau X^\alpha + t Y^\alpha + b^\alpha_{\beta\gamma}\, \tau X^\beta t Y^\gamma} - \underbrace{\tau X^\alpha}$

$+ b^\alpha_{\beta\gamma}\left(\tau X^\beta + t Y^\beta + b^\beta_{\mu\nu}\, \tau t X^\mu Y^\nu\right)(-\tau X)^\gamma$ + higher order

$= t Y^\alpha + t\tau\left[b^\alpha_{\beta\gamma} X^\beta Y^\gamma - b^\alpha_{\gamma\beta} X^\beta Y^\gamma \right]$ + other.

$I = \exp(\tau X)\exp(tY)\exp(-\tau X)$

$\left.\dfrac{\partial^2 I}{\partial t \partial \tau}\right|_{(0,0)} = [X\ Y]$ by Lemma

$= \left.\dfrac{d}{d\tau}\right|_{\tau=0} Ad_{\exp(\tau X)}(Y_e).$

$\Rightarrow$ by LHS & above computation

$[X\ Y]^\alpha = C^\alpha_{\beta\gamma} X^\beta Y^\gamma$

(iv) The oversight in the checking of the Jacobi identity.

$\varphi^\alpha(x, y) = \underbrace{x^\alpha + y^\alpha}_{1st} + \underbrace{b^\alpha_{\beta\gamma} x^\beta y^\gamma}_{2nd\ order} + h^\alpha_{ijk} x^i y^j y^k + g^\alpha_{ijk} x^i x^j y^k$

$$\Rightarrow \psi^\alpha(\psi(x,y), z) = \boxed{\psi^\alpha(x,y) +} z^\alpha + b^\alpha_{\beta\gamma}\psi^\beta(x,y) z^\gamma$$

$$+ h^\alpha_{ijk}\psi^i z^j z^k + g^\alpha_{ijk}\psi^i\psi^j z^k$$

$$= x^\alpha + y^\alpha + z^\alpha + b^\alpha_{\beta\gamma}\left(x^\beta + y^\beta + b^\beta_{st} x^s y^t + \cdots\right) z^\gamma$$

$$+ g^\alpha_{ijk}\left(x^i + y^i + \underset{+\cdots}{b^i_{st} x^s y^t}\right)\left(x^j + y^j + b^j_{st} x^s y^t\right) z^k$$

Hence $\exists$ $\boxed{b^\alpha_{\beta\gamma} b^\beta_{st}} x^s y^t z^\gamma$

$\boxed{g^\alpha_{st\gamma} x^s y^t z^\gamma + g^\alpha_{ts\gamma} x^s y^t z^\gamma}$ — extra terms.

Similarly $\psi^\alpha(x, \psi(y,z))$ has two extra terms involve

$h^\alpha_{st\gamma}$

One needs to include their contribution in checking the Jacobi identity.

$\varphi^i_j(u, x)$

$: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

(5) Proof of the theorem.

A PDE result:

$$\frac{\partial u^i}{\partial x_j \partial x_k} = \frac{\partial^2 u^i}{\partial x_k \partial x_j}$$

$$\begin{cases} \dfrac{\partial u^i}{\partial x^j} = \varphi^i_j(u_1, \dots, u_n, x_1, \dots, x_n) \\ u^i(x_1^0, \dots, x_n^0) = u_0^i \end{cases}$$

$u(x_0) = u_0$

$\Downarrow$ an unique solution

admits a ! solution if & only if

$$\boxed{\frac{\partial \varphi^i_k}{\partial u^\ell}\varphi^\ell_j + \frac{\partial \varphi^i_k}{\partial x^j} = \frac{\partial \varphi^i_j}{\partial u^\ell}\varphi^\ell_k + \frac{\partial \varphi^i_j}{\partial x^k}}$$

Compatibility Condition (Com) holds.

$\hookrightarrow$ Compatibility Conditions

$\Big[\ \varphi^i_j(u, x)$ are analytic, the solution is local near x^0

& analytic. Holds for smooth case. Can be reduced to Frobenius. cf. Narashimhan's Analysis on R & C manifolds.

The proof of the theorem involves a PDE & a ODE (system)

Using above

ⓐ The PDE part:

φ here is the inverse namely $\varphi(x) = x^{-1}$

$$v^\alpha_\beta(x) := \frac{\partial \psi^\alpha(x,y)}{\partial x^\beta}\bigg|_{y=\varphi(x)}$$

$$\Rightarrow \left\{ \begin{array}{l} \boxed{v^\alpha_\beta(\psi)\, \dfrac{\partial \psi^\beta(x,y)}{\partial x^\gamma} = v^\alpha_\gamma(x)} \\ \boxed{\psi(0,y) = y} \end{array} \right. \quad \text{(PDE-1)}$$

$$\boxed{\left\{ \begin{array}{l} \dfrac{\partial \psi^\beta}{\partial x^\gamma} = u^\beta_\alpha(\psi)\, v^\alpha_\gamma(x) \\ \psi(x_0) = y_0 \text{. Inverse of } v \end{array} \right.} \ \text{// } f^\beta_\gamma(\psi, x)$$

y is a parameter

$$\frac{\partial \psi^\beta}{\partial x^\gamma}(x,y)$$

The solution is

$$\boxed{\psi(x, x_0, y_0)}$$

Satisfies

$$\psi(x_0, x_0, y_0) = y_0$$

Let

$$\boxed{\psi(x,y) := \psi(x, e, y)}$$

$$\Rightarrow \psi(e,y) = y$$

$$\text{LHS} = \frac{\partial \psi^\alpha}{\partial x^\beta}\bigg|_{\substack{x=\psi(x,y) \\ y=\varphi(\psi(x,y))}}$$

$$\frac{\partial \psi^\alpha(\psi, z)}{\partial x^\beta} = \quad z = \varphi(\psi)$$

$$= \boxed{\frac{\partial \psi^\alpha(\psi(x,y), z)}{\partial x^\gamma}}\Bigg|_{z=\varphi(\psi(x,y))}$$

$$= \frac{\partial \psi^\alpha(x, \psi(y,z))}{\partial x^\gamma}\bigg|_{z=\varphi(\psi(x,y))} = \frac{\partial \psi^\alpha}{\partial x^\gamma}(x,y)\bigg|_{y=\varphi(x)} = v^\alpha_\gamma(x)$$

$z = \varphi(\psi(x,y))$

$$\psi(y,z) = y \cdot z = y\,(xy)^{-1} = x^{-1} = \varphi(x)$$

Hence the problem is reduce to solving (PDE-1), which is reduced to (Com)

The (Com) - condition: $\boxed{\dfrac{\partial \psi^\alpha}{\partial x^\gamma} = u^\alpha_\beta(\psi)\, v^\beta_\gamma(x)}$ — (PDE-1')

$\boxed{\varphi^\alpha_\gamma(\psi, x)} = \boxed{u^\alpha_\beta(\psi)\, v^\beta_\gamma(x)}$

The (Com) becomes the following:

$$\frac{\partial \varphi^\alpha_k}{\partial \psi^\beta}\varphi^\beta_j(\psi, x) + \frac{\partial \varphi^\alpha_k(\psi, x)}{\partial x^j}$$

$$= \frac{\partial u^\alpha_s}{\partial \psi^\beta} v^s_k(x) u^\beta_t(\psi)\, v^t_j(x) + u^\alpha_\beta(\psi)\frac{\partial v^\beta_k}{\partial x^j}$$

$$\parallel$$

$$\frac{\partial \varphi^\alpha_j}{\partial \psi^\beta}\varphi^\beta_k(\psi, x) + \frac{\partial \varphi^\alpha_j}{\partial x^k}(\psi, x)$$

$$= \frac{\partial u^\alpha_s}{\partial \psi^\beta} v^s_j(x)\, u^\beta_t(\psi)\, v^t_k(x) + u^\alpha_\beta(\psi)\frac{\partial v^\beta_j}{\partial x^k}$$

Multiply $v^i_\alpha(\psi)$ on both side & compute

$$-u^\alpha_s(\psi)\frac{\partial v^i_\alpha}{\partial \psi^\beta} u^\beta_t(\psi)\, v^s_k(x)\, v^t_j(x) + \frac{\partial v^i_k}{\partial x^j}$$

$$= -u^\alpha_s \frac{\partial v^i_\alpha}{\partial \psi^\beta} u^\beta_t(\psi)\, v^s_j(x)\, v^t_k(x) + \frac{\partial v^i_j}{\partial x^k}$$

$$\Rightarrow \quad \frac{\partial v^i_j}{\partial x^k} - \frac{\partial v^i_k}{\partial x^j} = \frac{\partial v^i_\alpha}{\partial \psi^\beta} u^\alpha_s(\psi)\, u^\beta_t(\psi)\, v^s_j(x)\, v^t_k(x)$$

$$- \frac{\partial v^i_\alpha}{\partial \psi^\beta} u^\alpha_s(\psi)\, u^\beta_t(\psi)\, v^s_k(x)\, v^t_j(x)$$

Moving terms

Multiply $u^j_\gamma(x)\, u^k_\delta(x)$ on both sides

$$\Rightarrow \left(\frac{\partial v^i_j(x)}{\partial x^k} - \frac{\partial v^i_k(x)}{\partial x^j}\right) u^j_\gamma(x)\, u^k_\delta(x)$$

$$= \frac{\partial v^i_\alpha}{\partial \psi^\beta}(\psi)\, u^\alpha_\gamma(\psi) u^\beta_\delta(\psi) - \frac{\partial v^i_\alpha}{\partial \psi^\beta}(\psi) u^\alpha_\delta(\psi) u^\beta_\gamma(\psi)$$

$$= \left(\frac{\partial v^i_\alpha}{\partial \psi^\beta}(\psi) - \frac{\partial v^i_\beta}{\partial \psi^\alpha}(\psi)\right) u^\alpha_\gamma\, u^\beta_\delta \quad \Rightarrow$$

As in seperation of variables

(Com) $\Leftrightarrow$ $\boxed{\sum_{j,k} \left(\frac{\partial v^i_j}{\partial x^k} - \frac{\partial v^i_k}{\partial x^j}\right) u^j_\gamma(x)\, u^k_\delta(x) \doteq \boxed{C^i_{\gamma\delta}}}$

A constant independent of x.

(Com) $\Leftrightarrow$ $\boxed{\left(\frac{\partial v^i_j}{\partial x^k} - \frac{\partial v^i_k}{\partial x^j}\right) = \sum_{\gamma\delta} C^i_{\gamma\delta}\, v^\gamma_j\, v^\delta_k}$

(b) Find $v^i_j(x)$ solves (Com)

This is reduced to solving ODE.

The Jacobi identity is the same as

$$[[\xi\ \eta], z] = C^s_{\beta\gamma} \xi^\beta \eta^\gamma z^t\, C^\alpha_{st} = C^\alpha_{st}\, C^s_{\beta\gamma}\, \xi^\beta \eta^\gamma z^t$$

$\xi = \xi^i e_i$ $\quad [e_i\ e_j] = C^k_{ij} e_k$

$\eta = \eta^j e_j$

$$\boxed{C_{st}^{\alpha} C_{\beta\gamma}^{s} + C_{s\beta}^{\alpha} C_{\gamma t}^{s} + C_{s\gamma}^{\alpha} C_{t\beta}^{s} = 0} \quad (J)$$

$(\beta\gamma, t) \to (\gamma t, \beta) \to (t\beta\gamma)$

This is needed to solve (Con) & find v_j^i.

PDE sometimes can be solved by ODEs, integrating along characteristics. But the method here is a bit more specialized since (J) is used crucially.

Sketch:

$$\begin{cases} \dfrac{dw_j^i}{dt} = \delta_j^i + C_{\alpha\beta}^{i} \boxed{a^{\alpha}} w_j^{\beta} \\ w_j^i(0) = 0 \end{cases} \qquad a \in \mathbb{R}^n$$

Solution called $w_j^i(t, a)$

Then let $\boxed{v_j^i(x) = w_j^i(1, x)}$.

It is a miracle that v_j^i solves (Con).

This fact heavily depends the Jacobi identity.

Summarize: $\{C_{jk}^i\}$ completely determined ψ, the product function near e. Hence the structure of G.

Reference on Lie's fundamental theorem namely, is Pontryagin Topological Groups. ch 10.

Checking Jacobi identity;

$$(b^{\alpha}_{\beta\gamma} - b^{\alpha}_{\gamma\beta})(b^{\beta}_{\mu\nu} - b^{\beta}_{\nu\mu}) + (b^{\alpha}_{\beta\nu} - b^{\alpha}_{\nu\beta})(b^{\beta}_{\gamma\mu} - b^{\beta}_{\mu\gamma}) + (b^{\alpha}_{\beta\mu} - b^{\alpha}_{\mu\beta})(b^{\beta}_{\nu\gamma} - b^{\beta}_{\gamma\nu})$$

$$= \underset{①}{b^{\alpha}_{\beta\gamma} b^{\beta}_{\mu\nu}} - \underset{④}{b^{\alpha}_{\beta\gamma} b^{\beta}_{\nu\mu}} + \underset{⑥}{b^{\alpha}_{\beta\nu} b^{\beta}_{\gamma\mu}} - \underset{③}{b^{\alpha}_{\beta\nu} b^{\beta}_{\mu\gamma}} + \underset{⑤}{b^{\alpha}_{\beta\mu} b^{\beta}_{\nu\gamma}} - \underset{②}{b^{\alpha}_{\beta\mu} b^{\beta}_{\gamma\nu}}$$

$$- \underset{⑥}{b^{\alpha}_{\gamma\beta} b^{\beta}_{\mu\nu}} + \underset{②}{b^{\alpha}_{\gamma\beta} b^{\beta}_{\nu\mu}} - \underset{⑤}{b^{\alpha}_{\nu\beta} b^{\beta}_{\gamma\mu}} + \underset{④}{b^{\alpha}_{\nu\beta} b^{\beta}_{\mu\gamma}} - \underset{①}{b^{\alpha}_{\mu\beta} b^{\beta}_{\nu\gamma}} + \underset{③}{b^{\alpha}_{\mu\beta} b^{\beta}_{\gamma\nu}}$$

$$(*) \rightarrow \boxed{b^{\alpha}_{\beta\gamma} b^{\beta}_{\mu\nu} - b^{\alpha}_{\mu\beta} b^{\beta}_{\nu\gamma} = \cancel{h^{\alpha}_{\mu\nu\gamma}} + \cancel{h^{\alpha}_{\mu\gamma\nu}} - \cancel{g^{\alpha}_{\mu\nu\gamma}} - g^{\alpha}_{\nu\mu\gamma}}$$

Apply permutation to the indices

①

$$-\left(b^{\alpha}_{\beta\mu} b^{\beta}_{\gamma\nu} - b^{\alpha}_{\gamma\beta} b^{\beta}_{\nu\mu}\right) = -\left(h^{\alpha}_{\gamma\nu\mu} + \cancel{h^{\alpha}_{\gamma\mu\nu}} - g^{\alpha}_{\gamma\nu\mu} - g^{\alpha}_{\nu\gamma\mu}\right)$$

$\mu \leftrightarrow \gamma$

$$-\left(b^{\alpha}_{\beta\nu} b^{\beta}_{\mu\gamma} - b^{\alpha}_{\mu\beta} b^{\beta}_{\gamma\nu}\right) = -\left(\cancel{h^{\alpha}_{\mu\gamma\nu}} + \cancel{h^{\alpha}_{\mu\nu\gamma}} - g^{\alpha}_{\mu\gamma\nu} - g^{\alpha}_{\gamma\mu\nu}\right)$$

$\nu \leftrightarrow \gamma$

$$-\left(b^{\alpha}_{\beta\gamma} b^{\beta}_{\nu\mu} - b^{\alpha}_{\nu\beta} b^{\beta}_{\mu\gamma}\right) = -\left(h^{\alpha}_{\nu\mu\gamma} + \cancel{h^{\alpha}_{\nu\gamma\mu}} - \cancel{g^{\alpha}_{\nu\mu\gamma}} - \cancel{g^{\alpha}_{\mu\nu\gamma}}\right)$$

$\mu \leftrightarrow \nu$

$$\begin{matrix}\gamma\mu\nu \\ \to \mu\nu\gamma\end{matrix} \quad b^{\alpha}_{\beta\mu} b^{\beta}_{\nu\gamma} - b^{\alpha}_{\nu\beta} b^{\beta}_{\gamma\mu} = \cancel{h^{\alpha}_{\nu\gamma\mu}} + h^{\alpha}_{\mu\nu\gamma} - g^{\alpha}_{\nu\gamma\mu} - g^{\alpha}_{\gamma\nu\mu}$$

$$\begin{matrix}\gamma\mu\nu \\ \mu\nu\gamma \\ \nu\gamma\mu\end{matrix} \quad b^{\alpha}_{\beta\nu} b^{\beta}_{\gamma\mu} - b^{\alpha}_{\gamma\beta} b^{\beta}_{\mu\gamma} = \cancel{h^{\alpha}_{\gamma\mu\nu}} + h^{\alpha}_{\gamma\nu\mu} - g^{\alpha}_{\gamma\mu\nu} - g^{\alpha}_{\mu\gamma\nu}$$

The right hand side all cancelled!

(1) $V^\alpha_\beta(x) = \left.\dfrac{\partial \psi^\alpha(x.\, y)}{\partial x^\beta}\right|_{y=\varphi(x)}$, namely $y = x^{-1}$

$$\psi^\alpha = x^\alpha + y^\alpha + b^\alpha_{\beta\gamma}\, x^\beta y^\gamma$$

$$\Rightarrow \quad \frac{\partial \psi^\alpha}{\partial x^\beta} = \delta^\alpha_\beta + b^\alpha_{\beta\gamma}\, y^\gamma$$

If $y = x^{-1}$ & $x = e \Rightarrow y = e$

$$\Rightarrow \quad \left.\frac{\partial \psi^\alpha}{\partial x^\beta}\right|_{y=\varphi(x)} = \delta^\alpha_\beta \quad \text{at} \quad x = e$$

Namely (V^α_β) is the identity matrix at $x = e$

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(2) $V^\alpha_\beta(\psi)\, \dfrac{\partial \psi^\beta}{\partial x^\gamma} = V^\alpha_\gamma(x) \iff \begin{cases} \dfrac{\partial \psi^\beta}{\partial x^\gamma} = U^\beta_\alpha(\psi)\, V^\alpha_\gamma(x) \\ \psi(x_0) = y_0 \end{cases}$

$\psi(0.\, y) = y$

$\psi(x)_{x_0.\, y_0}$ is the solution

$\Downarrow$

$\psi(x_0)_{x_0,\, y_0} = y_0$

Consider $\psi(x.\, y) = \psi(x)_{e,\, y}$

Then $\psi(e.\, y) = y$ by the

ψ so defined satisfies $\psi(x,\, \psi(y\, z)) = \psi(\psi(x\, y),\, z)$

$p = \psi(x.y) \qquad q = \psi(y.\, z)$

$w = \psi(p.\, z), \qquad w^* = \psi(x.\, q)$

$w(e) = \psi(\psi(e, y),\, z) = \psi(y\ z) = q$

$w^*(e) = q$

$$\frac{\partial w^\alpha}{\partial x^\gamma} = \frac{\partial w^\alpha}{\partial q^\beta}\frac{\partial q^\beta}{\partial x^\gamma} = U^\alpha_t(w)\, \underbrace{V^t_\beta(q)\, U^\beta_s(q)}_{=\delta_{ts}}\, V^s_\gamma(x) = U^\alpha_t(w)\, V^t_\gamma(x)$$

w^* & w are both solution of

$$\begin{cases} \dfrac{\partial w^\alpha}{\partial x^r} = u_t^\alpha(w)\, v_\gamma^t(x) \\ w(e) = v \end{cases} \qquad \text{(PDE-1)}$$

This verifies the associativity of ψ so-defined.

(3) why $w_j^i(1, x) = v_j^i(x)$ satisfies the ~~associativity~~. (com)

$$h_{jk}^i(t) = \frac{\partial w_k^i(t,a)}{\partial a^j} - \frac{\partial w_j^i(t,a)}{\partial a^k} - C_{\alpha\beta}^i\, w_j^\alpha(t,a)\, w_k^\beta(t,a)$$

the (Com) $\Leftrightarrow$ $h_{jk}^i(1) = 0$

We shall show $h_{jk}^i(t)$ satisfies a linear ODE & $=0$ at $t=0$

clearly at $t=0$ $h_{jk}^i(0) = 0$

Use the ODE $\Rightarrow$

$$\frac{d}{dt} h_{jk}^i(t) = C_{j\beta}^i w_k^\beta + C_{\alpha\beta}^i a^\alpha \frac{\partial w_k^\beta}{\partial a^j} - C_{k\beta}^i w_j^\beta - C_{\alpha\beta}^i a^\alpha \frac{\partial w_j^\beta}{\partial a^k}$$

$$- C_{\alpha\beta}^i\left(\delta_j^\alpha + C_{\gamma\delta}^\alpha a^\gamma w_j^\delta\right) w_k^\beta - C_{\alpha\beta}^i w_j^\alpha\left(\delta_k^\beta + C_{\gamma\delta}^\beta a^\gamma w_k^\delta\right)$$

Using the Jacobi $\Rightarrow$

$$\frac{dh_{jk}^i}{dt} = C_{\alpha\beta}^i a^\alpha\left(\frac{\partial w_k^\beta}{\partial a^j} - \frac{\partial w_j^\beta}{\partial a^k} - C_{\gamma\delta}^\beta w_j^\gamma w_k^\delta\right)$$

$$= C_{\alpha\beta}^i a^\alpha h_{jk}^\beta \Rightarrow \begin{cases} \dfrac{dh}{dt} = C_{\alpha\beta}\, a^\alpha h^\beta \\ h(0) = 0 \end{cases} \Rightarrow h(t) \equiv 0$$